

Queen Mary, University of London
MAS317 Linear Algebra II, 2007
Coursework 5

This coursework should be handed in by 4.00 pm on MONDAY October 29th in the orange box on the second floor in the mathematics building. Late coursework will not be accepted. Write your name and student number at the top and clip all the sheets together.

1. [20 marks] Let $U = \{(a, b, c) \in \mathbb{R}^3 : a + b - 3c = 0, a + 2b - 3c = 0\}$, and

$$W = \{(a, b, c) \in \mathbb{R}^3 : a - 3b - 5c = 0\}.$$

Find a basis for U and a basis for W .

With no further calculation answer the following.

- (a) Write down a spanning set for $U + W$.
 - (b) Is this a basis for $U + W$?
 - (c) Is $U + W$ a direct sum?
2. [40 marks] (i) Let
- $$U = \{(x_1, x_2, x_3, x_4, x_5) \mid x_1 - 2x_2 + x_4 = 0, x_3 - 2x_4 = 0\},$$
- and
- $$W = \{(x_1, x_2, x_3, x_4, x_5) \mid x_1 - 2x_2 + x_3 - x_4 = 0, x_2 + x_3 + x_5 = 0\}$$
- Find a basis for U and a basis for W . (See Note below.)
- What are the dimensions of these two spaces?
- (ii) Write $U \cap W$ in the same format as given for U and W , and find a basis for $U \cap W$. What are the dimensions of $U \cap W$ and $U + W$?
 - (iii) Use the Steinitz Exchange Lemma to find a basis for U consisting of your basis for $U \cap W$ and one or more vectors from your basis for U in (i).
Find a basis for W consisting of your basis for $U \cap W$ and one or more vectors from your basis for W in (i).
 - (iv) Write down a basis for $U + W$ containing bases for U and for W intersecting in a basis for $U \cap W$.

[For (iv), no further calculation is needed, and no proof is required. It has been done already in the proof of the Dimension Formula.]

[**Note:** the easiest way to write down a basis for the set of solutions of a system of linear equations **in echelon form** is as follows: If x_k is not a leading variable - ie not the first variable with non-zero coefficient in any of the equations, then work out a solution in which x_k is non-zero and all other non-leading variables are 0. These solutions are a basis for the space so its dimension is the number of non-leading variables ie the number of variables minus the number of non-zero equations.]

3. [10 marks]

Let V be a vector space of dimension 4, and let U and W be subspaces of V of dimensions 3 and 2 respectively. From the dimension formula, what are the possible dimensions of $U \cap W$?

4. [30 marks] (i) Let U , W_1 and W_2 be subspaces of a vector space V .

Prove that if $V = U \oplus W_1$ and $V = U \oplus W_2$ then W_1 and W_2 have the same dimension.

Prove that if the dimension of U is less than $(\dim V)/2$ then $W_1 \cap W_2 \neq \{0\}$.

(ii) In $\mathbb{R}^4$, let $U = \langle (1, 1, -1, 0), (3, 3, 0, -4) \rangle$.

Find a subspace W_1 of $\mathbb{R}^4$ such that $\mathbb{R}^4 = U \oplus W_1$.

Then find a subspace W_2 of $\mathbb{R}^4$ such that $\mathbb{R}^4 = U \oplus W_2$ and $W_1 \cap W_2 = \{0\}$.

Dr Sue McKay
19 October 2007